

Background field method in the gradient flow

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In perturbative consideration of the Yang–Mills gradient flow, it is useful to introduce a gauge non-covariant term (“gauge-fixing term”) to the flow equation that gives rise to a Gaussian damping factor also for gauge degrees of freedom. In the present paper, we consider a modified form of the gauge-fixing term that manifestly preserves covariance under the background gauge transformation. It is shown that our gauge-fixing term does not affect gauge-invariant quantities as the conventional gauge-fixing term. The formulation thus allows a background gauge covariant perturbative expansion of the flow equation that provides, in particular, a very efficient computational method of expansion coefficients in the small flow time expansion. The formulation can be generalized to systems containing fermions.
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Subject Index B01, B31, B32, B38

arXiv:1507.02360v3 [hep-lat] 24 Sep 2015

1. Introduction

As a novel method to define renormalized quantities, the Yang–Mills gradient flow [1, 2] and its extension to the fermion field [3] have attracted much attention in recent years, mainly in the context of lattice gauge theory. Reference [4] is a recent review, and Refs. [5–26] are more recent related studies.

Although the gradient flow in lattice gauge theory is utilized to study non-perturbative dynamics of gauge theory, information available through perturbative theory is always useful because the latter is well under analytic control. In the present paper, aiming at possible simplification in perturbative calculations associated with the gradient flow, we consider the application of the idea of the background field method [27–31] to the gradient flow. It is well known that this method considerably simplifies perturbative computation of, e.g., renormalization constants.

As clarified in Ref. [1], for perturbative consideration of the gradient flow, it is useful to introduce a “gauge-fixing term” that breaks gauge covariance of the flow equation; this term gives rise to a Gaussian damping factor also for gauge degrees of freedom and ensures a convergence property of momentum integrals. Here, we consider a modified form of the gauge-fixing term in the flow equation that manifestly preserves covariance under the background gauge transformation. It is shown that our gauge-fixing term does not affect gauge-invariant quantities, as the conventional gauge-fixing term. This formulation thus allows a background gauge covariant perturbative expansion of the flow equation that provides, in particular, a very efficient computational method of expansion coefficients in the small flow time expansion [2].

This paper is organized as follows. In Sect. 2, we present our general formulation. Both flow equations for the gauge field and for the fermion fields are considered. The most important observation is the independence of gauge-invariant quantities on the gauge-fixing term we introduce (Sect. 2.3). In subsequent sections, we consider applications of the formulation: In Sect. 3, we consider the computation of expansion coefficients in the small flow time expansion [2] relevant to the construction of the lattice energy–momentum tensor; this computation was carried out in Ref. [32] using a cumbersome diagrammatic method. We observe that the application of our formulation provides a very efficient non-diagrammatic computational method, that is quite analogous to that of Ref. [33], for the expansion coefficients.¹ In Sect. 4, we consider the small flow time expansion relevant to the construction of the axial-vector current [34]. The last section is devoted to the conclusion.

Here is a summary of our notation: Our generators T^a of the gauge group G are anti-Hermitian and the structure constants are defined by $[T^a, T^b] = f^{abc}T^c$. Quadratic Casimirs are defined by $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$ and, for a representation R , $\text{tr}_R(T^a T^b) = -T(R)\delta^{ab}$ and $T^a T^a = -C_2(R)1$. We also denote $\text{tr}_R(1) = \dim(R)$. For example, for the fundamental N representation of $SU(N)$ for which $\dim(N) = N$, the conventional choice is

$$C_2(SU(N)) = N, \quad T(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}. \quad (1.1)$$

¹Unfortunately, the results of this new simple computational scheme do not coincide with the results in Ref. [32], revealing that there are errors in the one-loop diagrammatic calculation in Ref. [32]. The diagrams in which the mistakes were made in Ref. [32] have been completely identified. For corrected results, see the errata for Refs. [32, 35] and Refs. [36, 37].

Our gamma matrices are Hermitian and for the trace over the spinor index we set $\text{tr}(1) = 4$ for any spacetime dimension D . The chiral matrix is defined by $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ for any D and thus

$$\text{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \begin{cases} 4\epsilon_{\mu\nu\rho\sigma}, & \mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise}, \end{cases} \quad (1.2)$$

where the totally anti-symmetric tensor is normalized as $\epsilon_{0123} = 1$.

2. Flow equations with a background covariant gauge

2.1. Gradient flow equation with a background covariant gauge

The gradient flow for the gauge potential is defined by [1]

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \quad B_\mu(t=0, x) = A_\mu(x), \quad (2.1)$$

where $t \in [0, \infty)$, and

$$D_\mu = \partial_\mu + [B_\mu, \cdot], \quad G_{\mu\nu}(t, x) = \partial_\mu B_\nu(t, x) - \partial_\nu B_\mu(t, x) + [B_\mu(t, x), B_\nu(t, x)] \quad (2.2)$$

denote the covariant derivative and the field strength of the flowed gauge field, respectively. The last term in the first relation of Eq. (2.1) breaks gauge covariance and here it is referred to as a “gauge-fixing term”. As noted in Ref [1], for perturbative consideration of the gradient flow, such as that in Ref. [2], it is useful to introduce such a gauge-breaking term because it gives rise to a Gaussian damping factor also for gauge degrees of freedom and ensures a convergence property of momentum integrals. It can, however, be shown that [1] any gauge-invariant quantity, that does not contain the flow time derivative ∂_t , is independent of the “gauge parameter” α_0 and physical observables are not affected by the gauge-fixing term.

In the present paper, we propose a slight modification of the gauge-fixing term in Eq. (2.1). First, following the general idea of the background field method [27–31], we decompose the original gauge potential into the background part $\hat{A}_\mu(x)$ and the quantum part $a_\mu(x)$ as

$$A_\mu(x) = \hat{A}_\mu(x) + a_\mu(x). \quad (2.3)$$

We also decompose the flowed gauge potential $B(t, x)$ into the background part $\hat{B}_\mu(t, x)$ and the quantum part $b_\mu(t, x)$ as

$$B_\mu(t, x) = \hat{B}_\mu(t, x) + b_\mu(t, x). \quad (2.4)$$

Then, our proposal is to adopt, instead of Eq. (2.1),

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t, x), \quad B_\mu(t=0, x) = A_\mu(x), \quad (2.5)$$

where

$$\hat{D}_\mu = \partial_\mu + [\hat{B}_\mu, \cdot] \quad (2.6)$$

are the covariant derivatives with respect to the *background flowed field*.

As a further natural assumption, we suppose that the background field is evolved by its own flow equation:

$$\partial_t \hat{B}_\mu(t, x) = \hat{D}_\nu \hat{G}_{\nu\mu}(t, x), \quad \hat{B}_\mu(t=0, x) = \hat{A}_\mu(x), \quad (2.7)$$

where

$$\hat{G}_{\mu\nu}(t, x) = \partial_\mu \hat{B}_\nu(t, x) - \partial_\nu \hat{B}_\mu(t, x) + [\hat{B}_\mu(t, x), \hat{B}_\nu(t, x)] \quad (2.8)$$

is the field strength of the background field.

2.2. Covariance under the background gauge transformation

The original gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + D_\mu \omega(x), \quad (2.9)$$

may be decomposed into the background part and the quantum part; how this decomposition is made is the heart of the background field method [27–31]. A fundamental notion is the *background gauge transformation*, defined by

$$\hat{A}_\mu(x) \rightarrow \hat{A}_\mu(x) + \hat{D}_\mu \omega(x), \quad a_\mu(x) \rightarrow a_\mu(x) + [a_\mu(x), \omega(x)]. \quad (2.10)$$

The sum of these two reproduces the original gauge transformation (2.9). Under this background gauge transformation, the quantum gauge field transforms as the adjoint representation. This transformation can also be generalized to the flowed fields as²

$$\hat{B}_\mu(t, x) \rightarrow \hat{B}_\mu(t, x) + \hat{D}_\mu \omega(x), \quad b_\mu(t, x) \rightarrow b_\mu(t, x) + [b_\mu(t, x), \omega(x)]. \quad (2.11)$$

Note that here we are assuming that the transformation function $\omega(x)$ does not depend on the flow time t .

Since \hat{D}_μ in Eq. (2.6) transforms covariantly under the background gauge transformation (2.11), our flow equation (2.5) transforms covariantly under the background gauge transformation; fields transformed by the background gauge transformation obey the identical equation. This is the reason for our choice of the particular gauge-fixing term in Eq. (2.5) instead of the conventional one in Eq. (2.1).

Now let us confirm that our gauge-fixing term in Eq. (2.5) does not affect gauge-invariant quantities.

2.3. Independence of gauge-invariant quantities of the gauge parameter α_0

Although our gauge-fixing term $\alpha_0 D_\mu \hat{D}_\nu b_\nu(t, x)$ in Eq. (2.5) differs from the conventional one in Eq. (2.1), one can still see that any gauge-invariant quantity, that does not contain the flow time derivative ∂_t is independent of the “gauge parameter” α_0 ; the gauge-fixing term thus does not affect gauge-invariant quantities.

To see this, we consider the following “quantum gauge transformation”³

$$\hat{B}_\mu(t, x) \rightarrow \hat{B}_\mu(t, x), \quad b_\mu(t, x) \rightarrow b_\mu(t, x) + D_\mu \omega(t, x), \quad (2.12)$$

whose transformation function $\omega(t, x)$ *does* depend on the flow time t . Note that the sum of these two reproduces the original gauge transformation (2.9) with $\omega(x) \rightarrow \omega(t, x)$. Under

² The covariant derivative \hat{D}_μ in the first relation is defined with respect to the flowed background field $\hat{B}_\mu(t, x)$.

³ Here the covariant derivative D_μ in the second expression is defined with respect to the flowed field $B_\mu(t, x)$.

this infinitesimal transformation, we have

$$\partial_t B_\mu(t, x) \rightarrow \partial_t B_\mu(t, x) + [\partial_t B_\mu(t, x), \omega(t, x)] + D_\mu \partial_t \omega(t, x), \quad (2.13)$$

$$D_\nu G_{\nu\mu}(t, x) \rightarrow D_\nu G_{\nu\mu}(t, x) + [D_\nu G_{\nu\mu}(t, x), \omega(t, x)], \quad (2.14)$$

$$\hat{D}_\nu b_\nu(t, x) \rightarrow \hat{D}_\nu b_\nu(t, x) + \hat{D}_\nu D_\nu \omega(t, x), \quad (2.15)$$

and

$$D_\mu \hat{D}_\nu b_\nu(t, x) \rightarrow D_\mu \hat{D}_\nu b_\nu(t, x) + [D_\mu \hat{D}_\nu b_\nu(t, x), \omega(t, x)] + D_\mu D_\nu \hat{D}_\nu \omega(t, x), \quad (2.16)$$

where in deriving the last expression we have noted

$$\hat{D}_\nu D_\nu \omega(t, x) = D_\nu \hat{D}_\nu \omega(t, x) + [\hat{D}_\nu b_\nu(t, x), \omega(t, x)]. \quad (2.17)$$

From these expressions, we see that under Eq. (2.12), the flow equation (2.5) changes to

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t, x) - D_\mu (\partial_t - \alpha_0 D_\nu \hat{D}_\nu) \omega(t, x). \quad (2.18)$$

This shows that, by choosing $\omega(t, x)$ as a solution of

$$(\partial_t - \alpha_0 D_\nu \hat{D}_\nu) \omega(t, x) = -\delta \alpha_0 \hat{D}_\nu b_\nu(t, x), \quad \omega(t=0, x) = 0, \quad (2.19)$$

the transformed flowed field (that has the same initial value as the original one) obeys the flow equation (2.5) with $\alpha_0 \rightarrow \alpha_0 + \delta \alpha_0$. Since a gauge-invariant quantity that does not contain the t derivative is invariant under Eq. (2.12), this shows that such a gauge-invariant quantity is independent of α_0 . Physical observables are not affected by the gauge fixing term in Eq. (2.5); the introduction of the gauge-fixing term is thus a physically allowed modification of the flow equation.

2.4. Classical perturbative solution to the flow equation

Now, using Eq. (2.7) in Eq. (2.5), we have the flow equation for the quantum field:

$$\partial_t b_\mu(t, x) = \left[\delta_{\mu\nu} \hat{D}^2 + (\alpha_0 - 1) \hat{D}_\mu \hat{D}_\nu \right] b_\nu(t, x) + 2[\hat{G}_{\mu\nu}(t, x), b_\nu(t, x)] + \hat{R}_\mu(t, x), \quad (2.20)$$

where

$$\begin{aligned} \hat{R}_\mu(t, x) \equiv & 2[b_\nu(t, x), \hat{D}_\nu b_\mu(t, x)] - [b_\nu(t, x), \hat{D}_\mu b_\nu(t, x)] \\ & + (\alpha_0 - 1)[b_\mu(t, x), \hat{D}_\nu b_\nu(t, x)] + [b_\nu(t, x), [b_\nu(t, x), b_\mu(t, x)]]. \end{aligned} \quad (2.21)$$

The adjoint actions in these expressions can conveniently be expressed in terms of matrix multiplication, if one introduces the adjoint representation. We thus define

$$\hat{B}_\mu^{ab}(t, x) \equiv \hat{B}_\mu^c(t, x) f^{acb}, \quad (2.22)$$

$$\hat{\mathcal{D}}_\mu^{ab} \equiv \delta^{ab} \partial_\mu + \hat{B}_\mu^{ab}(t, x), \quad (2.23)$$

$$\hat{\mathcal{G}}_{\mu\nu}^{ab}(t, x) \equiv \hat{G}_{\mu\nu}^c(t, x) f^{acb}. \quad (2.24)$$

With these notations, the flow equation for the quantum field reads

$$\partial_t b_\mu^a(t, x) = \left[\delta_{\mu\nu} \hat{D}^2 + (\alpha_0 - 1) \hat{\mathcal{D}}_\mu \hat{\mathcal{D}}_\nu + 2\hat{\mathcal{G}}_{\mu\nu}(t, x) \right]^{ab} b_\nu^b(t, x) + \hat{R}_\mu^a(t, x), \quad (2.25)$$

where

$$\begin{aligned}\hat{R}_\mu^a(t, x) = & 2f^{abc}b_\nu^b(t, x)\hat{\mathcal{D}}_\nu^{cd}b_\mu^d(t, x) - f^{abc}b_\nu^b(t, x)\hat{\mathcal{D}}_\mu^{cd}b_\nu^d(t, x) \\ & + (\alpha_0 - 1)f^{abc}b_\mu^b(t, x)\hat{\mathcal{D}}_\nu^{cd}b_\nu^d(t, x) + f^{abc}f^{cde}b_\nu^b(t, x)b_\nu^d(t, x)b_\mu^e(t, x).\end{aligned}\quad (2.26)$$

A formal solution to Eq. (2.25) is then given by

$$b_\mu^a(t, x) = \int d^4y \left[\hat{K}_t^{ab}(x, y)_{\mu\nu} a_\nu^b(y) + \int_0^t ds \hat{K}_{t-s}^{ab}(x, y)_{\mu\nu} \hat{R}_\nu^b(s, y) \right], \quad (2.27)$$

where the heat kernel $\hat{K}_t^{ab}(x, y)_{\mu\nu}$ is defined as an object that satisfies

$$\partial_t \hat{K}_t^{ab}(x, y)_{\mu\nu} = \left[\delta_{\mu\lambda} \hat{\mathcal{D}}^2 + (\alpha_0 - 1) \hat{\mathcal{D}}_\mu \hat{\mathcal{D}}_\lambda + 2\hat{\mathcal{G}}_{\mu\lambda}(t, x) \right]^{ac} \hat{K}_t^{cb}(x, y)_{\lambda\nu}, \quad (2.28)$$

$$\hat{K}_{t=0}^{ab}(x, y)_{\mu\nu} = \delta^{ab} \delta_{\mu\nu} \delta(x - y). \quad (2.29)$$

The heat kernel defined by Eqs. (2.28) and (2.29) may be expressed in the form of a time-ordered product containing the flowed background field. If one is considering a particular situation in which the background field $\hat{A}(x)$ can be assumed to obey the Yang–Mills equation of motion,

$$\hat{D}_\nu \hat{F}_{\nu\mu}(x) = 0, \quad (2.30)$$

then Eq. (2.7) implies that the background gauge field does not flow:

$$\hat{B}(t, x) = \hat{A}(x). \quad (2.31)$$

Then the heat kernel in the “Feynman gauge” $\alpha_0 = 1$ can be written, suppressing the gauge and Lorentz indices, in a very compact form:

$$\begin{aligned}\hat{K}_t(x, y) = & T \exp \left\{ \int_0^t ds \left[\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{G}}(s, x) \right] \right\} \delta(x - y) \\ = & e^{t[\hat{\mathcal{D}}_x^2 + 2\hat{\mathcal{F}}(x)]} \delta(x - y).\end{aligned}\quad (2.32)$$

In the last expression, the covariant derivative is defined with respect to the background gauge field at vanishing flow time, $\hat{A}_\mu(x)$; we have also introduced the corresponding field strength in the adjoint representation,

$$\hat{\mathcal{F}}_{\mu\nu}^{ab}(x) \equiv \hat{F}_{\mu\nu}^c(x) f^{acb}. \quad (2.33)$$

In the application to the small flow time expansion in the next section, we can assume Eq. (2.30) without loss of generality. We can then use Eq. (2.32) for the heat kernel which greatly simplifies the computation.

2.5. Tree-level propagator of the flowed gauge field

So far we have considered the flow equation (2.20) at the classical level. The quantum field at vanishing flow time, $a_\nu^b(y)$, contained in Eq. (2.27) is actually the subject of the functional

integral with the Boltzmann weight, specified by the Yang–Mills action

$$S = \frac{1}{4g_0^2} \int d^4x F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) \quad (2.34)$$

and the gauge-fixing term in the background gauge [27–31]

$$S_{\text{gauge-fixing}} = \frac{\lambda_0}{2g_0^2} \int d^4x \hat{D}_\mu a_\mu^a(x) \hat{D}_\nu a_\nu^a(x), \quad (2.35)$$

which also preserves covariance under the background gauge transformation. Then, in the presence of the background field, the action quadratic in the quantum field is given by

$$\begin{aligned} & (S + S_{\text{gauge-fixing}})|_{O(a^2)} \\ &= -\frac{1}{2g_0^2} \int d^4x a_\mu^a(x) \left[\delta_{\mu\nu} \hat{D}^2 + (\lambda_0 - 1) \hat{D}_\mu \hat{D}_\nu + 2\hat{\mathcal{F}}_{\mu\nu}(x) \right]^{ab} a_\nu^b(x), \end{aligned} \quad (2.36)$$

and thus the tree-level propagator in the Feynman gauge $\lambda_0 = 1$ is written as

$$\left\langle a_\mu^a(x) a_\nu^b(y) \right\rangle_0 = g_0^2 \left(\frac{-1}{\hat{D}_x^2 + 2\hat{\mathcal{F}}(x)} \right)_{\mu\nu}^{ab} \delta(x - y). \quad (2.37)$$

If one can further assume Eq. (2.30) for the background field, then the heat kernel (in the Feynman gauge) is given by Eq. (2.32). Then, from Eq. (2.27), the tree-level propagator of the flowed quantum field, in the presence of the background field, is given by

$$\begin{aligned} \left\langle b_\mu^a(t, x) b_\nu^b(s, y) \right\rangle_0 &= g_0^2 \left(e^{t[\hat{D}_x^2 + 2\hat{\mathcal{F}}(x)]} \frac{-1}{\hat{D}_x^2 + 2\hat{\mathcal{F}}(x)} \right)_{\mu\rho}^{ac} \left(e^{s[\hat{D}_y^2 + 2\hat{\mathcal{F}}(y)]} \right)_{\nu\rho}^{bc} \delta(x - y) \\ &= g_0^2 \left(e^{(t+s)[\hat{D}_x^2 + 2\hat{\mathcal{F}}(x)]} \frac{-1}{\hat{D}_x^2 + 2\hat{\mathcal{F}}(x)} \right)_{\mu\nu}^{ab} \delta(x - y), \end{aligned} \quad (2.38)$$

where in the last equality we have noted

$$\left[\hat{D}_y^2 + 2\hat{\mathcal{F}}(y) \right]_{\mu\nu}^{ab} \delta(x - y) = \left[\hat{D}_x^2 + 2\hat{\mathcal{F}}(x) \right]_{\nu\mu}^{ba} \delta(x - y). \quad (2.39)$$

The above expression (2.38), which is manifestly covariant under the background gauge transformation, will be fully employed in our application to the small flow time expansion in the next section.

2.6. Fermion flow

We can also consider the “background covariant gauge” in the flow of fermion fields [3]; we adopt the following flow equations:

$$\partial_t \chi(t, x) = \left\{ D^2 - \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\} \chi(t, x), \quad \chi(t = 0, x) = \psi(x), \quad (2.40)$$

$$\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) \left\{ \overleftarrow{D}^2 + \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\}, \quad \bar{\chi}(t = 0, x) = \bar{\psi}(x), \quad (2.41)$$

where the covariant derivatives on the fermion fields are defined by

$$D_\mu = \partial_\mu + B_\mu, \quad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - B_\mu, \quad (2.42)$$

and

$$\hat{D}_\mu = \partial_\mu + \hat{B}_\mu, \quad \hat{\overleftarrow{D}}_\mu \equiv \overleftarrow{\partial}_\mu - \hat{B}_\mu. \quad (2.43)$$

On the other hand, in these expressions and in what follows, $[\hat{D}_\mu b_\mu(t, x)]$ stands for the background covariant derivative on the quantum gauge field, defined in Eq. (2.6).

One can again see that the gauge parameter α_0 is irrelevant for gauge-invariant quantities with our gauge-fixing terms in Eqs. (2.40) and (2.41). To see this, we again consider the infinitesimal transformation, Eq. (2.12), and

$$\chi(t, x) \rightarrow [1 - \omega(t, x)] \chi(t, x), \quad \bar{\chi}(t, x) \rightarrow \bar{\chi}(t, x) [1 + \omega(t, x)]. \quad (2.44)$$

Then using Eq. (2.15), after some calculation, we see that the flow equations are changed as

$$\partial_t \chi(t, x) = \left\{ D^2 - \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\} \chi(t, x) + (\partial_t - \alpha_0 D_\mu \hat{D}_\mu) \omega(t, x) \chi(t, x), \quad (2.45)$$

$$\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) \left\{ \overleftarrow{D}^2 + \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\} - \bar{\chi}(t, x) (\partial_t - \alpha_0 D_\mu \hat{D}_\mu) \omega(t, x). \quad (2.46)$$

These show that again by choosing $\omega(t, x)$ as the solution to Eq. (2.19), we can shift α_0 to $\alpha_0 + \delta\alpha_0$. Gauge-invariant quantities (that do not contain the flow time derivative) are hence not affected by the gauge-fixing terms in Eqs. (2.40) and (2.41).

We also decompose the fermion fields into the background part and the quantum part as

$$\chi(t, x) = \hat{\chi}(t, x) + k(t, x), \quad \bar{\chi}(t, x) = \hat{\bar{\chi}}(t, x) + \bar{k}(t, x). \quad (2.47)$$

$$\psi(x) = \hat{\psi}(x) + p(x), \quad \bar{\psi}(x) = \hat{\bar{\psi}}(x) + \bar{p}(x), \quad (2.48)$$

and assume that the background fields themselves are evolved according to

$$\partial_t \hat{\chi}(t, x) = \hat{D}^2 \hat{\chi}(t, x), \quad \hat{\chi}(t=0, x) = \hat{\psi}(x), \quad (2.49)$$

$$\partial_t \hat{\bar{\chi}}(t, x) = \hat{\bar{\chi}}(t, x) \overleftarrow{\hat{D}}^2, \quad \hat{\bar{\chi}}(t=0, x) = \hat{\bar{\psi}}(x). \quad (2.50)$$

Then, from Eqs. (2.40) and (2.41), the quantum fields obey the flow equations

$$\begin{aligned} \partial_t k(t, x) &= \left\{ D^2 - \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\} k(t, x) \\ &\quad + \left\{ (1 - \alpha_0) [\hat{D}_\mu b_\mu(t, x)] + 2b_\mu(t, x) \hat{D}_\mu + b_\mu(t, x) b_\mu(t, x) \right\} \hat{\chi}(t, x), \\ k(t=0, x) &= p(x), \end{aligned} \quad (2.51)$$

$$\begin{aligned} \partial_t \bar{k}(t, x) &= \bar{k}(t, x) \left\{ \overleftarrow{D}^2 + \alpha_0 [\hat{D}_\mu b_\mu(t, x)] \right\} \\ &\quad + \hat{\bar{\chi}}(t, x) \left\{ -(1 - \alpha_0) [\hat{D}_\mu b_\mu(t, x)] - 2\overleftarrow{\hat{D}}_\mu b_\mu(t, x) + b_\mu(t, x) b_\mu(t, x) \right\}, \\ \bar{k}(t=0, x) &= \bar{p}(x). \end{aligned} \quad (2.52)$$

If we further assume that the background gauge field fulfills Eq. (2.30), the background gauge field does not evolve as Eq. (2.31) and we can write down relatively simple expressions for the solution of the fermion flow. The solution to the flow equations (2.49) and (2.50) can be expressed as

$$\hat{\chi}(t, x) = e^{t\hat{D}^2} \hat{\psi}(x), \quad \hat{\bar{\chi}}(t, x) = \hat{\bar{\psi}}(x) e^{t\overleftarrow{\hat{D}}^2}. \quad (2.53)$$

Then, the solution to the flow equations (2.51) and (2.52) is given by

$$k(t, x) = e^{t\hat{D}^2} p(x) + \int_0^t ds e^{(t-s)\hat{D}^2} \left[2b_\mu(s, x) \hat{D}_\mu + b_\mu(s, x) b_\mu(s, x) \right] \left[e^{s\hat{D}^2} \hat{\psi}(x) + k(s, x) \right], \quad (2.54)$$

$$\bar{k}(t, x) = \bar{p}(x) e^{t\overleftarrow{\hat{D}}^2} + \int_0^t ds \left[\hat{\bar{\psi}}(x) e^{s\overleftarrow{\hat{D}}^2} + \bar{k}(s, x) \right] \left[-2\overleftarrow{\hat{D}}_\mu b_\mu(s, x) + b_\mu(s, x) b_\mu(s, x) \right] e^{(t-s)\overleftarrow{\hat{D}}^2}, \quad (2.55)$$

where we have adopted the ‘‘Feynman gauge’’ $\alpha_0 = 1$ for simplicity.

The quantum fields at vanishing flow time, $p(x)$ and $\bar{p}(x)$, are subjects of the functional integral with the conventional action,

$$\begin{aligned} S &= \int d^D x \bar{\psi}(x)(\not{D} + m_0)\psi(x) \\ &= \int d^D x \left[\hat{\bar{\psi}}(x) + \bar{p}(x) \right] \left(\hat{\not{D}} + \not{\phi} + m_0 \right) \left[\hat{\psi}(x) + p(x) \right]. \end{aligned} \quad (2.56)$$

Thus the tree-level propagator of quantum fermion fields, in the presence of the background gauge field, is given by

$$\langle p(x)\bar{p}(y) \rangle_0 = \frac{1}{\hat{\not{D}}_x + m_0} \delta(x - y). \quad (2.57)$$

3. Application: Small flow time expansion relevant to the energy–momentum tensor

As noted in Ref. [2], any local composite operator of flowed fields can be expressed as, in the limit of $t \rightarrow 0$, an asymptotic series of local composite operators of fields at vanishing flow time. In Ref. [32], this *small flow time expansion* (with use of perturbation theory) was exploited to construct a universal formula for the energy–momentum tensor. This formula with lattice regularization was then numerically tested in Ref. [38] by applying it to the bulk thermodynamics of quenched QCD. The universal formula can be generalized to general vector-like gauge theories [35] and to various asymptotically free theories [39–41]. In Refs. [42, 43], application of the gradient flow to the lattice energy–momentum tensor is studied from a somewhat different perspective.

In the present paper, we restrict ourselves to the case of the pure Yang–Mills theory [32] and consider the small flow time expansion in the form,⁴

$$\begin{aligned} G_{\mu\rho}^a(t, x)G_{\nu\sigma}^a(t, x) \\ \stackrel{t \rightarrow 0}{\sim} \langle G_{\mu\rho}^a(t, x)G_{\nu\sigma}^a(t, x) \rangle + \zeta_{11}(t)F_{\mu\rho}^a(x)F_{\nu\sigma}^a(x) + \zeta_{12}(t)\delta_{\mu\nu}F_{\rho\sigma}^a(x)F_{\rho\sigma}^a(x) + O(t), \end{aligned} \quad (3.1)$$

where the $O(t)$ term is the contribution of composite operators of the mass dimension being equal to or greater than six. Because of symmetry, only the above two four-dimensional operators can appear on the right-hand side. The expansion coefficients can be evaluated in perturbation theory and, to the one-loop order, we write

$$\zeta_{11}(t) = 1 + \zeta_{11}^{(1)}(t) + \cdots, \quad \zeta_{12}(t) = 0 + \zeta_{12}^{(1)}(t) + \cdots. \quad (3.2)$$

If these coefficients are obtained in the dimensional regularization (with the spacetime dimension $D = 4 - 2\epsilon$), then the correctly normalized conserved energy–momentum tensor (with

⁴In Appendix B, we compute the small flow time expansion of an operator corresponding to the topological density—another gauge-invariant dimension-four operator.

the vacuum expectation value subtracted) can be written as [32, 35]

$$\begin{aligned} \{T_{\mu\nu}\}_R(x) = \lim_{t \rightarrow 0} \Big\{ c_1(t) \left[G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) - \frac{1}{4} \delta_{\mu\nu} G_{\rho\sigma}^a(t, x) G_{\rho\sigma}^a(t, x) \right] \right. \\ \left. + c_2(t) \left[\delta_{\mu\nu} G_{\rho\sigma}^a(t, x) G_{\rho\sigma}^a(t, x) - \langle \delta_{\mu\nu} G_{\rho\sigma}^a(t, x) G_{\rho\sigma}^a(t, x) \rangle \right] \right\}, \end{aligned} \quad (3.3)$$

where

$$c_1(t) = \frac{1}{g_0^2} \left[1 - \zeta_{11}^{(1)}(t) \right], \quad c_2(t) = \frac{1}{g_0^2} \left[-\frac{1}{2} \epsilon \zeta_{12}^{(1)}(t) \right]. \quad (3.4)$$

Since bare composite operators of the flowed gauge field are automatically renormalized operators [2], the formula (3.3) should hold irrespective of an adopted regularization,⁵ in this sense the formula is universal. In particular, it should hold with lattice regularization with which the construction of a correctly normalized conserved energy–momentum tensor is not straightforward. It is thus of great interest to compute the expansion coefficients in Eq. (3.1). As we will see below, the background field method we have developed provides a very efficient non-diagrammatic computational method of the expansion coefficients (at least in the one-loop level).

Now, to determine the expansion coefficients $\zeta_{11}(t)$ and $\zeta_{12}(t)$ in Eq. (3.1), we consider 1PI diagrams containing the composite operators $G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x)$ and $F_{\mu\rho}^a(x) F_{\nu\rho}^a(x)$ with external lines of the background gauge field $\hat{B}_\mu(t, x)$ only (i.e., no external line of the quantum field). In the tree level, since the flow time evolution is purely classical,

$$\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \rangle_{\text{1PI}} \stackrel{t \rightarrow 0}{\sim} \hat{F}_{\mu\rho}^a(x) \hat{F}_{\nu\rho}^a(x) + O(t), \quad (3.5)$$

$$\langle F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \rangle_{\text{1PI}} = \hat{F}_{\mu\rho}^a(x) \hat{F}_{\nu\rho}^a(x). \quad (3.6)$$

Comparing these two relations, we find

$$G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \stackrel{t \rightarrow 0}{\sim} F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) + O(t). \quad (3.7)$$

This gives the tree-level contributions in Eq. (3.2).

Next, we consider one-loop 1PI diagrams containing the composite operators with external lines of the background gauge field. Such 1PI diagrams can be obtained, by taking the contraction of quantum fields in the expansion of the composite operators by the propagator in the presence of the background field. The expansion of the composite operator $G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x)$ in the quadratic order yields

$$\begin{aligned} G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \Big|_{O(b^2)} \\ = (\delta_{\mu\alpha} \delta_{\nu\delta} \delta_{\beta\gamma} - \delta_{\mu\alpha} \delta_{\nu\gamma} \delta_{\beta\delta} - \delta_{\mu\beta} \delta_{\nu\delta} \delta_{\alpha\gamma} + \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\delta}) \left[\hat{\mathcal{D}}_\alpha b_\beta(t, x) \right]^a \left[\hat{\mathcal{D}}_\delta b_\gamma(t, x) \right]^a \\ - b_\nu(t, x) \hat{\mathcal{F}}_{\mu\rho}^a(x) b_\rho(t, x) - b_\mu(t, x) \hat{\mathcal{F}}_{\nu\rho}^a(x) b_\rho(t, x). \end{aligned} \quad (3.8)$$

At this stage we note that to read off the expansion coefficients in Eq. (3.1) as Eq. (3.7), we can assume that the background field satisfies the Yang–Mills equation of motion, Eq (2.30), because $\hat{F}_{\mu\rho}^a(x) \hat{F}_{\nu\rho}^a(x)$ does not vanish under the equation of motion. Then the background

⁵ The coefficients $c_1(t)$ and $c_2(t)$ in Eq. (3.4) become finite for $\epsilon \rightarrow 0$ when expressed in terms of renormalized quantities; see below.

field does not flow $\hat{B}(t, x) = \hat{A}(x)$ and we can use the simple expression (2.38) for the propagator. The contraction then yields

$$\begin{aligned}
& \left\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \right|_{O(b^2)} \Big\rangle_{\text{1PI}} \\
&= g_0^2 (\delta_{\mu\alpha} \delta_{\nu\delta} \delta_{\beta\gamma} - \delta_{\mu\alpha} \delta_{\nu\gamma} \delta_{\beta\delta} - \delta_{\mu\beta} \delta_{\nu\delta} \delta_{\alpha\gamma} + \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\delta}) \hat{\mathcal{D}}_\alpha^{ab} \left(e^{2t\hat{\Delta}} \frac{1}{\hat{\Delta}} \right)_{\beta\gamma}^{bc} \hat{\mathcal{D}}_\delta^{ca} \delta(x-y)|_{y=x} \\
&+ g_0^2 \hat{\mathcal{F}}_{\mu\rho}^{ab}(x) \left(e^{2t\hat{\Delta}} \frac{1}{\hat{\Delta}} \right)_{\rho\nu}^{ba} \delta(x-y)|_{y=x} + g_0^2 \hat{\mathcal{F}}_{\nu\rho}^{ab}(x) \left(e^{2t\hat{\Delta}} \frac{1}{\hat{\Delta}} \right)_{\rho\mu}^{ba} \delta(x-y)|_{y=x}, \quad (3.9)
\end{aligned}$$

where

$$\hat{\Delta} \equiv \hat{\mathcal{D}}^2 + 2\hat{\mathcal{F}}, \quad (3.10)$$

and we have noted

$$\hat{\mathcal{D}}_{y\delta}^{ac} \delta(x-y) = -\hat{\mathcal{D}}_{x\delta}^{ca} \delta(x-y). \quad (3.11)$$

However, as explained in Refs. [37] (see also Ref. [39]), instead of the 1PI function (3.9) itself, it is much convenient to consider the difference

$$\left\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \right|_{O(b^2)} - F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \Big|_{O(a^2)} \Big\rangle_{\text{1PI}}, \quad (3.12)$$

because possible infrared divergences are cancelled out in this combination. The one-loop 1PI function, which contains $F_{\mu\rho}^a(x) F_{\nu\rho}^a(x)$ is given by simply setting $t = 0$ in Eq. (3.9). Then the difference can be expressed as an integral over an auxiliary variable ξ as,

$$\begin{aligned}
& \left\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \right|_{O(b^2)} - F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \Big|_{O(a^2)} \Big\rangle_{\text{1PI}} \\
&= 2g_0^2 \int_0^t d\xi \left[(\delta_{\mu\alpha} \delta_{\nu\delta} \delta_{\beta\gamma} - \delta_{\mu\alpha} \delta_{\nu\gamma} \delta_{\beta\delta} - \delta_{\mu\beta} \delta_{\nu\delta} \delta_{\alpha\gamma} + \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\delta}) \hat{\mathcal{D}}_\alpha^{ab} \left(e^{2\xi\hat{\Delta}} \right)_{\beta\gamma}^{bc} \hat{\mathcal{D}}_\delta^{ca} \right. \\
&\quad \left. + \hat{\mathcal{F}}_{\mu\rho}^{ab}(x) \left(e^{2\xi\hat{\Delta}} \right)_{\rho\nu}^{ba} + \hat{\mathcal{F}}_{\nu\rho}^{ab}(x) \left(e^{2\xi\hat{\Delta}} \right)_{\rho\mu}^{ba} \right] \delta(x-y)|_{y=x}. \quad (3.13)
\end{aligned}$$

In this expression, it is obvious that there is no infrared divergence, because derivative operators appear only in positive powers.

We then set

$$\delta(x-y) = \int \frac{d^D p}{(2\pi)^D} e^{ipx} e^{-ipy}, \quad (3.14)$$

and moves the plain wave e^{ipx} the most left-hand side, by noting

$$\hat{\mathcal{D}}_\mu e^{ipx} = e^{ipx} (ip_\mu + \hat{\mathcal{D}}_\mu), \quad (3.15)$$

as usual in the calculation of anomalies in the path integral [33, 44–47]. After the rescaling of the integration variables, $p_\mu \rightarrow p_\mu/\sqrt{\xi}$, we have

$$\begin{aligned}
& \left\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \Big|_{O(b^2)} - F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \Big|_{O(a^2)} \right\rangle_{\text{1PI}} \\
&= 2g_0^2 \int_0^t d\xi \xi^{-D/2} \int \frac{d^D p}{(2\pi)^D} e^{-2p^2} \\
&\quad \times \text{tr} \left[(\delta_{\mu\alpha} \delta_{\nu\delta} \delta_{\beta\gamma} - \delta_{\mu\alpha} \delta_{\nu\gamma} \delta_{\beta\delta} - \delta_{\mu\beta} \delta_{\nu\delta} \delta_{\alpha\gamma} + \delta_{\mu\beta} \delta_{\nu\gamma} \delta_{\alpha\delta}) \right. \\
&\quad \times \xi^{-1} \left(ip_\alpha + \sqrt{\xi} \hat{\mathcal{D}}_\alpha \right) \left(e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}} \right)_{\beta\gamma} \left(ip_\delta + \sqrt{\xi} \hat{\mathcal{D}}_\delta \right) \\
&\quad \left. + \hat{\mathcal{F}}_{\mu\rho}(x) \left(e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}} \right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x) \left(e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}} \right)_{\rho\mu} \right], \quad (3.16)
\end{aligned}$$

where the trace tr is for the gauge indices. It is interesting to note that all the information of relevant one-loop 1PI diagrams is contained in this single compact expression; in a conventional calculational scheme [32], on the other hand, one has to compute at least 12 1PI diagrams to obtain the expansion coefficients in Eq. (3.1).

Now, since we are interested in the small flow time limit $t \rightarrow 0$ of Eq. (3.16) and since $\xi \in [0, t]$, we may expand the integrand with respect to ξ . For $t \rightarrow 0$, only terms to $O(\xi^{-D/2+1})$ under the integral can give rise to non-vanishing contributions for $D \rightarrow 4$. The expansion of the combination $e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}}$ to $O(\xi^2)$ is given in Appendix A. Although the remaining algebraic calculation after the Gaussian integration over p , by noting

$$[\hat{\mathcal{D}}_\rho, \hat{\mathcal{D}}_\sigma] = \hat{\mathcal{F}}_{\rho\sigma} \quad (3.17)$$

is somewhat lengthy, it is rather straightforward. In this calculation, it is quite helpful to note that the final expression for Eq. (3.16) must be symmetric under $\mu \leftrightarrow \nu$ by definition; we may thus simply discard any terms anti-symmetric under $\mu \leftrightarrow \nu$. In this way, we finally arrive at

$$\begin{aligned}
& \left\langle G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \Big|_{O(b^2)} - F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \Big|_{O(a^2)} \right\rangle_{\text{1PI}} \\
& \stackrel{t \rightarrow 0}{\sim} \frac{g_0^2}{(4\pi)^2} \dim(G) \frac{3}{8t^2} \delta_{\mu\nu} \\
& \quad + \frac{g_0^2}{(4\pi)^2} \left[\frac{11}{3} \epsilon(t)^{-1} + \frac{7}{3} \right] \text{tr} [\hat{\mathcal{F}}(x)^2]_{\mu\nu} \\
& \quad + \frac{g_0^2}{(4\pi)^2} \left[-\frac{11}{12} \epsilon(t)^{-1} - \frac{1}{6} \right] \delta_{\mu\nu} \text{tr} [\hat{\mathcal{F}}(x)^2]_{\rho\rho} + O(t), \quad (3.18)
\end{aligned}$$

where

$$\epsilon(t)^{-1} \equiv \frac{1}{\epsilon} + \ln(8\pi t). \quad (3.19)$$

Since

$$\text{tr} [\hat{\mathcal{F}}(x)^2]_{\mu\nu} = \hat{\mathcal{F}}_{\mu\rho}^{ab}(x) \hat{\mathcal{F}}_{\rho\nu}^{ba}(x) = f^{acb} f^{bda} \hat{F}_{\mu\rho}^c(x) \hat{F}_{\rho\nu}^d(x) = C_2(G) \hat{F}_{\mu\rho}^a(x) \hat{F}_{\nu\rho}^a(x), \quad (3.20)$$

recalling the tree-level relations (3.5) and (3.6), Eq. (3.18) shows that the small flow time expansion to the one-loop order is given by

$$\begin{aligned}
& G_{\mu\rho}^a(t, x) G_{\nu\rho}^a(t, x) \\
& \stackrel{t \rightarrow 0}{\sim} \frac{g_0^2}{(4\pi)^2} \dim(G) \frac{3}{8t^2} \delta_{\mu\nu} \\
& + \left\{ 1 + \frac{g_0^2}{(4\pi)^2} C_2(G) \left[\frac{11}{3} \epsilon(t)^{-1} + \frac{7}{3} \right] \right\} F_{\mu\rho}^a(x) F_{\nu\rho}^a(x) \\
& + \frac{g_0^2}{(4\pi)^2} C_2(G) \left[-\frac{11}{12} \epsilon(t)^{-1} - \frac{1}{6} \right] \delta_{\mu\nu} F_{\rho\sigma}^a(x) F_{\rho\sigma}^a(x) + O(t). \tag{3.21}
\end{aligned}$$

From this, $\zeta_{11}^{(1)}$ and $\zeta_{12}^{(1)}$ in Eq. (3.2) are given by

$$\zeta_{11}^{(1)}(t) = \frac{g_0^2}{(4\pi)^2} C_2(G) \left[\frac{11}{3} \epsilon(t)^{-1} + \frac{7}{3} \right], \tag{3.22}$$

$$\zeta_{12}^{(1)}(t) = \frac{g_0^2}{(4\pi)^2} C_2(G) \left[-\frac{11}{12} \epsilon(t)^{-1} - \frac{1}{6} \right], \tag{3.23}$$

and then Eq. (3.4) gives $c_1(t)$ and $c_2(t)$ in Eq. (3.3). In terms of the renormalized gauge coupling g in the $\overline{\text{MS}}$ scheme,

$$\frac{1}{g_0^2} = \frac{1}{g^2} + b_0 \left(\frac{1}{\epsilon} - \ln \mu^2 \right), \tag{3.24}$$

where b_0 is the one-loop coefficient in the beta function,

$$b_0 = \frac{1}{(4\pi)^2} C_2(G) \frac{11}{3}, \tag{3.25}$$

we have

$$c_1(t) = \frac{1}{g^2} - b_0 \ln(8\pi\mu^2 t) - \frac{1}{(4\pi)^2} C_2(G) \frac{7}{3}, \tag{3.26}$$

$$c_2(t) = \frac{1}{8} b_0. \tag{3.27}$$

The above non-diagrammatic one-loop computation of coefficients $c_1(t)$ and $c_2(t)$ is much simpler and quicker than the diagrammatic calculation carried out in Ref. [32]. Unfortunately, the results of the above calculation do not coincide with the results in Ref. [32], revealing that there are errors in the one-loop diagrammatic calculation in Ref. [32].⁶

⁶ In particular, Eq. (4.30) and (4.31) of Ref. [32] should be

$$c_1 = \ln \sqrt{\pi} + \frac{7}{22} \simeq 0.890547, \tag{3.28}$$

$$c_2 = \ln \sqrt{\pi} - \frac{7}{44} + \frac{b_1}{2b_0^2} \simeq 0.834762. \tag{3.29}$$

Equations (4.32) and (4.33) of Ref. [35] should be replaced by Eqs. (3.22) and (3.23), respectively, and consequently, Eq. (4.72) of Ref. [35] should be

$$c_1(t) = \frac{1}{\bar{g}(1/\sqrt{8t})^2} - b_0 \ln \pi - \frac{1}{(4\pi)^2} \left[\frac{7}{3} C_2(G) - \frac{3}{2} T(R) N_f \right], \tag{3.30}$$

4. Application: Small flow time expansion of the axial-vector current

As another application of the present formulation, we consider the small flow time expansion of the axial-vector current of the flowed fermion fields [34]:

$$\bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) \stackrel{t \rightarrow 0}{\sim} \left[1 + \xi^{(1)}(t) \right] \bar{\psi}(x) \gamma_\mu \gamma_5 t^A \psi(x) + O(t), \quad (4.1)$$

where t^A is the generator of the flavor symmetry group and $\xi^{(1)}(t)$ is the expansion coefficient at the one-loop level. Because of symmetry, only the axial-vector current at vanishing flow time can appear as the leading $O(t^0)$ term in the right-hand side. To find $\xi^{(1)}(t)$, we set the background gauge field to zero and consider one-loop 1PI diagrams containing the composite operator $\bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x)$ with external lines of the background fermion fields, $\hat{\psi}(t, x)$ and $\hat{\bar{\psi}}(t, x)$ (no external line of the quantum fields).

As Eqs. (3.5) and (3.6), at the tree level,

$$\langle \bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) \rangle_{\text{1PI}} \stackrel{t \rightarrow 0}{\sim} \hat{\bar{\psi}}(x) \gamma_\mu \gamma_5 t^A \hat{\psi}(x) + O(t), \quad (4.2)$$

$$\langle \bar{\psi}(x) \gamma_\mu \gamma_5 t^A \psi(x) \rangle_{\text{1PI}} = \hat{\bar{\psi}}(x) \gamma_\mu \gamma_5 t^A \hat{\psi}(x). \quad (4.3)$$

For the one-loop level, again as Eq. (3.12), it is convenient to consider the difference:

$$\langle \bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) - \bar{\psi}(x) \gamma_\mu \gamma_5 t^A \psi(x) \rangle_{\text{1PI}}, \quad (4.4)$$

because this combination is free from infrared divergences. For the first term of Eq. (4.4), by using the decomposition (2.47), we have

$$\begin{aligned} & \bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) \\ &= \hat{\bar{\chi}}(t, x) \gamma_\mu \gamma_5 t^A \hat{\chi}(t, x) + \hat{\bar{\chi}}(t, x) \gamma_\mu \gamma_5 t^A k(t, x) + \bar{k}(t, x) \gamma_\mu \gamma_5 t^A \hat{\chi}(t, x) + \bar{k}(t, x) \gamma_\mu \gamma_5 t^A k(t, x). \end{aligned} \quad (4.5)$$

We can then use Eqs. (2.54) and (2.55) to express the quantum flowed fields, $k(t, x)$ and $\bar{k}(t, x)$, in terms of fermion fields at vanishing flow time. Equation (4.1) shows that to find the coefficient $\xi^{(1)}(t)$, we may set the background fermion fields to be constant which makes the calculation quite easy. Then, as terms which contribute to one-loop 1PI diagrams with external lines of background fermion fields, we have

$$k(t, x) = e^{t\partial^2} p(x) + \int_0^t ds e^{(t-s)\partial^2} b_\mu(s, x) b_\mu(s, x) \hat{\psi} + \int_0^t ds e^{(t-s)\partial^2} 2b_\mu(s, x) \partial_\mu e^{s\partial^2} p(x), \quad (4.6)$$

$$\begin{aligned} & \bar{k}(t, x) \\ &= \bar{p}(x) e^{t\overleftarrow{\partial}^2} + \int_0^t ds \hat{\bar{\psi}} b_\mu(s, x) b_\mu(s, x) e^{(t-s)\overleftarrow{\partial}^2} + \int_0^t ds \bar{p}(x) e^{s\overleftarrow{\partial}^2} (-2) \overleftarrow{\partial}_\mu b_\mu(s, x) e^{(t-s)\overleftarrow{\partial}^2}. \end{aligned} \quad (4.7)$$

where $\bar{g}(1/\sqrt{8t})$ is the running gauge coupling in the $\overline{\text{MS}}$ scheme at the renormalization scale $\mu = 1/\sqrt{8t}$. The expressions just below Eqs. (5) and (6) of Ref. [38] should be

$$\bar{s}_1 = \frac{7}{22} + \frac{1}{2} \gamma_E - \ln 2 \simeq -0.0863575299274, \quad (3.31)$$

$$\bar{s}_2 = \frac{21}{44} - \frac{b_1}{2b_0^2} = \frac{27}{484} \simeq 0.0557851239669. \quad (3.32)$$

The erratum for Ref. [38] will appear soon.

In Eqs. (4.6) and (4.7), the quantum fields at vanishing flow time, $p(x)$ and $\bar{p}(x)$, are subject to the functional integral with the action (2.56). Through the interaction terms in Eq. (2.56), $p(x)$ and $\bar{p}(x)$ become the background fields, $\hat{\psi}$ and $\hat{\bar{\psi}}$. Considering the contraction by the propagator (2.57) in $\langle p(x)(-1) \int d^D y \bar{p}(y) \phi(y) \hat{\psi} \rangle$ and $\langle (-1) \int d^D y \hat{\bar{\psi}} \phi(y) p(y) \bar{p}(x) \rangle$,⁷ this effect of interaction vertices can effectively be represented by the substitutions,

$$p(x) \rightarrow - \int d^D y \frac{1}{\not{\partial}_x + m_0} \delta(x-y) \phi(y) \hat{\psi}, \quad (4.8)$$

$$\bar{p}(x) \rightarrow - \int d^D y \hat{\bar{\psi}} \phi(y) \frac{1}{\not{\partial}_y + m_0} \delta(y-x). \quad (4.9)$$

Note that these substitutions accompany a gauge interaction vertex coming from the action (2.56) because we are considering 1PI diagrams. Then the contraction of the quantum gauge fields in the expectation value of Eq. (4.5) by the propagator (2.38) is very simple. In this way, we have the one-loop expression for the first term of Eq. (4.4). Then, by simply setting $t = 0$ in that expression, we have the one-loop expression for the second term of Eq. (4.4). The resulting difference is free from infrared divergences and under the dimensional regularization, it is simple to obtain at the one-loop level

$$\begin{aligned} & \langle \bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) - \bar{\psi}(x) \gamma_\mu \gamma_5 t^A \psi(x) \rangle_{\text{1PI}}, \\ & \stackrel{t \rightarrow 0}{\sim} \frac{g_0^2}{(4\pi)^2} C_2(R) (-3) \left[\epsilon(t)^{-1} + \frac{7}{6} \right] \hat{\bar{\psi}}(x) \gamma_\mu \gamma_5 t^A \hat{\psi}(x) + O(g_0^4). \end{aligned} \quad (4.10)$$

Because of the tree-level relations (4.2) and (4.3), Eq. (4.10) shows that

$$\bar{\chi}(t, x) \gamma_\mu \gamma_5 t^A \chi(t, x) \stackrel{t \rightarrow 0}{\sim} \left\{ 1 + \frac{g_0^2}{(4\pi)^2} C_2(R) (-3) \left[\epsilon(t)^{-1} + \frac{7}{6} \right] \right\} \bar{\psi}(x) \gamma_\mu \gamma_5 t^A \psi(x), \quad (4.11)$$

which coincides with the result in Ref. [34].

As discussed in Ref. [34], Eq. (4.11) shows that the correctly normalized axial-vector current can be expressed as

$$j_{5\mu}^A(x) = \lim_{t \rightarrow 0} \left\{ 1 + \frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2} C_2(R) \left[-\frac{1}{2} + \ln(432) \right] \right\} \hat{\bar{\chi}}(t, x) \gamma_\mu \gamma_5 t^A \hat{\chi}(t, x), \quad (4.12)$$

where [35]

$$\hat{\bar{\chi}}(t, x) = \sqrt{\frac{-2 \dim(R) N_f}{(4\pi)^2 t^2 \langle \bar{\chi}(t, x) \overleftrightarrow{D} \chi(t, x) \rangle}} \bar{\chi}(t, x), \quad (4.13)$$

$$\hat{\chi}(t, x) = \sqrt{\frac{-2 \dim(R) N_f}{(4\pi)^2 t^2 \langle \bar{\chi}(t, x) \overleftrightarrow{D} \chi(t, x) \rangle}} \chi(t, x), \quad (4.14)$$

(N_f is the number of flavors) and

$$\overleftrightarrow{D}_\mu \equiv D_\mu - \overleftarrow{D}_\mu. \quad (4.15)$$

⁷ Note that we are now setting the background gauge field to zero.

5. Conclusion

In the present paper, we have developed a background field method (or a background gauge covariant gauge fixing, more appropriately) for the gradient flow equations. This formulation allows a manifestly background gauge covariant perturbative expansion of the flow equations. We illustrated the power of the method by applying it to the one-loop calculation of expansion coefficients in the small flow time expansion relevant to the energy–momentum tensor. This new simple computational scheme revealed that there were errors in the old diagrammatic calculation in Ref. [32] (the errors have been identified and corrected [36, 37]).

Since our method provides a greatly simplified computational scheme for known one-loop computations, we can expect that it can also be useful in more complicated situations, such as the two-loop computation of the expansion coefficients. We hope to come back to possible further applications of the present formulation in the near future.

Acknowledgments

The author would like to thank Kazuo Fujikawa and Hiroki Makino for helpful discussions. The work of H. S. is supported in part by Grant-in-Aid for Scientific Research 23540330.

A. Expansion of $e^{4i\sqrt{\xi}p\cdot\hat{D}+2\xi\hat{\Delta}}$

A straightforward expansion yields

$$\begin{aligned}
& \left(e^{4i\sqrt{\xi}p\cdot\hat{D}+2\xi\hat{\Delta}} \right)_{\mu\nu} \\
&= \delta_{\mu\nu} + 4ip \cdot \hat{D} \delta_{\mu\nu} \xi^{1/2} + \left[2\hat{\Delta}_{\mu\nu} - 8(p \cdot \hat{D})^2 \delta_{\mu\nu} \right] \xi \\
&\quad + \left[4i\hat{\Delta}_{\mu\nu} p \cdot \hat{D} + 4ip \cdot \hat{D} \hat{\Delta}_{\mu\nu} - \frac{32}{3} i(p \cdot \hat{D})^3 \delta_{\mu\nu} \right] \xi^{3/2} \\
&\quad + \left[2\hat{\Delta}_{\mu\nu}^2 - \frac{16}{3} \hat{\Delta}_{\mu\nu} (p \cdot \hat{D})^2 - \frac{16}{3} p \cdot \hat{D} \hat{\Delta}_{\mu\nu} p \cdot \hat{D} - \frac{16}{3} (p \cdot \hat{D})^2 \hat{\Delta}_{\mu\nu} + \frac{32}{3} (p \cdot \hat{D})^4 \delta_{\mu\nu} \right] \xi^2 \\
&\quad + O(\xi^{5/2}), \tag{A1}
\end{aligned}$$

where

$$\hat{\Delta}_{\mu\nu} = \hat{D}^2 \delta_{\mu\nu} + 2\hat{\mathcal{F}}_{\mu\nu}, \tag{A2}$$

$$\hat{\Delta}_{\mu\nu}^2 = \hat{D}^2 \hat{D}^2 \delta_{\mu\nu} + 2\hat{D}^2 \hat{\mathcal{F}}_{\mu\nu} + 2\hat{\mathcal{F}}_{\mu\nu} \hat{D}^2 + 4\hat{\mathcal{F}}_{\mu\nu}^2. \tag{A3}$$

B. Small flow time expansion of the topological density

In this appendix, we present the small flow time expansion of an operator corresponding to the topological charge density in the one-loop order. First, to the quadratic order in the quantum field, we have

$$\begin{aligned}
& \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \Big|_{O(b^2)} \\
&= 2\epsilon_{\mu\nu\rho\sigma} \left\{ 2 \left[\hat{D}_\mu b_\nu(t, x) \right]^a \left[\hat{D}_\rho b_\sigma(t, x) \right]^a - b_\mu(t, x) \hat{\mathcal{F}}_{\nu\rho}(x) b_\sigma(t, x) \right\}. \tag{B1}
\end{aligned}$$

The contraction by the propagator (2.38) then yields

$$\begin{aligned} & \left\langle \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \Big|_{O(b^2)} \right\rangle_{\text{1PI}} \\ &= -2g_0^2 \epsilon_{\mu\nu\rho\sigma} \left[2\hat{\mathcal{D}}_\mu^{ab} \left(e^{2t\hat{\Delta}} \frac{1}{\hat{\Delta}} \right)_{\nu\rho}^{bc} \hat{\mathcal{D}}_\sigma^{ca} + \hat{\mathcal{F}}_{\mu\nu}^{ab}(x) \left(e^{2t\hat{\Delta}} \frac{1}{\hat{\Delta}} \right)_{\rho\sigma}^{ba} \right] \delta(x-y)|_{y=x}. \end{aligned} \quad (\text{B2})$$

The same procedure as led to Eq. (3.16) in the main text then gives rise to

$$\begin{aligned} & \left\langle \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \Big|_{O(b^2)} - \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) \Big|_{O(a^2)} \right\rangle_{\text{1PI}}, \\ &= -4g_0^2 \epsilon_{\mu\nu\rho\sigma} \int_0^t d\xi \xi^{-D/2} \int \frac{d^D p}{(2\pi)^D} e^{-2p^2} \\ & \quad \times \text{tr} \left[2\xi^{-1} \left(ip + \sqrt{\xi} \hat{\mathcal{D}} \right)_\mu \left(e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}} \right)_{\nu\rho} \left(ip + \sqrt{\xi} \hat{\mathcal{D}} \right)_\sigma + \hat{\mathcal{F}}_{\mu\nu}(x) \left(e^{4i\sqrt{\xi} p \cdot \hat{\mathcal{D}} + 2\xi \hat{\Delta}} \right)_{\rho\sigma} \right]. \end{aligned} \quad (\text{B3})$$

The expansion with respect to ξ is much simpler than Eq. (3.16). Thus we give some details of the calculation for illustration.

First, in the integrand of Eq. (B3), any term that is symmetric under the exchange of indices $\{\mu, \nu, \rho, \sigma\}$ does not contribute because of $\epsilon_{\mu\nu\rho\sigma}$. Then, using Eq. (A1), it is easy to see that the expansion of Eq. (B3) to $O(\xi^{-D/2+1})$ in the integrand yields

$$\begin{aligned} & 16g_0^2 \epsilon_{\mu\nu\rho\sigma} \int_0^t d\xi \xi^{-D/2+1} \int \frac{d^D p}{(2\pi)^D} e^{-2p^2} \\ & \quad \times \text{tr} \left\{ 4p_\mu \left[\hat{\mathcal{F}}_{\nu\rho}(x) p \cdot \hat{\mathcal{D}} + p \cdot \hat{\mathcal{D}} \hat{\mathcal{F}}_{\nu\rho}(x) \right] \hat{\mathcal{D}}_\sigma + 4\hat{\mathcal{D}}_\mu \left[\hat{\mathcal{F}}_{\nu\rho}(x) p \cdot \hat{\mathcal{D}} + p \cdot \hat{\mathcal{D}} \hat{\mathcal{F}}_{\nu\rho}(x) \right] p_\sigma \right. \\ & \quad \left. - 2\hat{\mathcal{D}}_\mu \hat{\mathcal{F}}_{\nu\rho}(x) \hat{\mathcal{D}}_\sigma - \hat{\mathcal{F}}_{\mu\nu}(x) \hat{\mathcal{F}}_{\rho\sigma}(x) \right\}. \end{aligned} \quad (\text{B4})$$

After the momentum integrations,

$$\int \frac{d^D p}{(2\pi)^D} e^{-2p^2} \left\{ \begin{array}{c} 1 \\ p_\mu p_\nu \end{array} \right\} = \frac{1}{(8\pi)^{D/2}} \left\{ \begin{array}{c} 1 \\ \frac{1}{4} \delta_{\mu\nu} \end{array} \right\}, \quad (\text{B5})$$

Eq. (B4) becomes

$$\frac{16}{(8\pi)^{D/2}} g_0^2 \epsilon_{\mu\nu\rho\sigma} \int_0^t d\xi \xi^{-D/2+1} \text{tr} \left[\hat{\mathcal{F}}_{\nu\rho}(x) \hat{\mathcal{D}}_\mu \hat{\mathcal{D}}_\sigma + \hat{\mathcal{D}}_\mu \hat{\mathcal{D}}_\sigma \hat{\mathcal{F}}_{\nu\rho}(x) - \hat{\mathcal{F}}_{\mu\nu}(x) \hat{\mathcal{F}}_{\rho\sigma}(x) \right]. \quad (\text{B6})$$

Finally, using Eq. (3.17), we see that this combination identically vanishes. We infer that, therefore, in the pure Yang–Mills theory,

$$\epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \stackrel{t \rightarrow 0}{\sim} (1 + 0 \cdot g_0^2) \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) + O(t), \quad (\text{B7})$$

to the one-loop order.

It turns out that, from very general grounds,

$$\epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) \stackrel{t \rightarrow 0}{\sim} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) + O(t) \quad (\text{B8})$$

holds in the pure Yang–Mills theory in *all orders of perturbation theory* [25]. To see this, one first notes [48]

$$\partial_t [\epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x)] = \partial_\mu W_\mu(t, x), \quad W_\mu(t, x) = 4\epsilon_{\mu\nu\rho\sigma} D_\lambda G_{\lambda\nu}^a(t, x) G_{\rho\sigma}^a(t, x), \quad (\text{B9})$$

where $W_\mu(t, x)$ is a gauge-invariant dimension 5 axial-vector operator. This shows that

$$\epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a(t, x) G_{\rho\sigma}^a(t, x) = \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) + \partial_\mu \int_0^t dt' W_\mu(t', x). \quad (\text{B10})$$

We then consider the small flow time expansion of $W_\mu(t', x)$ in the last term. Since there is no gauge-invariant axial vector of dimension < 5 in the pure Yang–Mills theory, the small flow-time expansion of $W_\mu(t', x)$ starts from a dimension 5 operator with an $O(t'^0)$ coefficient (possibly with logarithmic corrections). This implies that the last term of Eq. (B10) is $O(t)$ and thus Eq. (B8). Our explicit one-loop calculation (B7) is consistent with this general property (B8), as it should be.

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